Scientific report on the implementation of the project PN-III-P4-ID-PCE-2016-0065 during 2017

During this period 6 scientific papers were elaborated, 4 among them being accepted or published in ISI-ranked journals and 2 papers are under review. The content of these 6 papers, that cover completely the objectives proposed for this period, can be synthesized as follows:

1. Liviu Ornea, Vladimir Slesar : *The spectral sequence of the canonical foliation of a Vaisman manifold*, Ann. Glob. Anal. Geom., DOI 10.1007/s10455-017-9579-8, published electronically November 2017.

In this paper we investigate the spectral sequence associated with a Riemannian foliation which arises naturally on a Vaisman manifold. Using the Betti numbers of the underlying manifold we establish a lower bound for the dimension of some terms of this cohomological object. This way we obtain cohomological obstructions for two-dimensional foliations to be induced from a Vaisman structure. We show that if the foliation is quasi-regular the lower bound is realized. In the final part of the paper we discuss two examples.

2. V. Slesar, M. Visinescu, G.E. Vîlcu, *Hidden Symmetries in Sasaki-Einstein Geometries*, Phys. Atom. Nucl., Vol. 80, No. 4, 2017, pp. 801-807.

We describe a method for constructing Killing-Yano tensors on Sasaki spaces using their geometrical properties, without the need of solving intricate generalized Killing equations. We obtain the Killing-Yano tensors on toric Sasaki-Einstein spaces using the fact that the metric cones of these spaces are Calabi-Yau manifolds which in turn are described in terms of toric data. We extend the search of Killing-Yano tensors on mixed 3-Sasakian manifolds. We illustrate the method by explicit construction of Killing forms on some spaces of current interest.

3. A.D. Vîlcu, G.E. Vîlcu, *An algorithm to estimate the vertices of a tetrahedron from uniform random points inside*, Ann. Mat. Pura Appl., DOI:10.1007/s10231-017-0688-6, published electronically August 2017.

In this paper, we give an algorithm to infer the positions of the vertices of an unknown tetrahedron, given a sample of points which are uniformly distributed within the tetrahedron. The accuracy of the algorithm is demonstrated by applying a Monte Carlo simulation technique.

4. S. Dăscălescu, C. Năstăsescu, L. Năstăsescu, *Graded semisimple algebras are symmetric*, Journal of Algebra 491 (2017), 207-218.

Frobenius algebras are algebraic objects that arose from representation theory of groups, but they are also present in other areas of mathematics, for example in the theory of Hopf algebras and quantum groups, in the theory of compact oriented manifolds, in topological quantum field theory, etc. A finite dimensional algebra A over a field k is Frobenius if A and its k-dual A^* are isomorphic as left A-modules, or equivalently, as right A-modules. There are certain Frobenius algebras having more symmetry and also a rich representation theory; these are the symmetric algebras. A is called symmetric if A and A^* are isomorphic as A, A-bimodules, or equivalently, if there exists a linear map $\lambda : A \to k$ such that $\lambda(ab) = \lambda(ba)$ for any $a, b \in A$, and Ker λ does not contain non-zero left ideals. It was proved by Eilenberg and Nakayama in their pioneering work on Frobenius algebras that a finite dimensional semisimple algebra is symmetric. Frobenius algebras can also be considered in arbitrary monoidal categories, while symmetric algebras can be defined in monoidal categories with more structure, for example in sovereign categories. It is an interesting problem to understand the structure of Frobenius (symmetric) algebras in certain monoidal categories. A case of special interest is a category of vector spaces graded by an arbitrary group G. A finite dimensional G-graded algebra $A = \oplus_{g\in G} A_g$ is symmetric in this category (we shortly say that A is graded symmetric), if A and A^* are isomorphic as G-graded A, A-bimodules. Here the grading on A^* is given by $(A^*)_g = \{f \in A^* \mid f(A_h) = 0 \text{ for any } h \neq g^{-1}\}$. Our main result is

Theorem A. *A finite dimensional graded semisimple algebra is graded symmetric.*

The first step is to look at graded division algebras. We make some general considerations about graded symmetric graded crossed products, and we apply them to graded division algebras. If A is a finite dimensional G-graded division algebra, and D is the homogeneous component of degree e of A, where e is the neutral element of G, we construct an action of G on the space $D/[D, D]$, and prove that A is graded symmetric if and only if $D/[D, D]$ has non-zero coinvariants. By using Hilbert Theorem 90 we show that $D/[D, D]$ is isomorphic as a kG-module to the center ℓ of D, which is itself a kG-module. Finally, ℓ has non-zero coinvariants by using the Normal Basis Theorem. Next we transfer the result to graded simple algebras using the graded version of Wedderburn's Theorem, and further to graded semisimple algebras.

On the other hand we show that the center of a graded division algebra need not be symmetric. However, we prove the following.

Theorem B. *Let A be a finite dimensional G-graded division algebra such that* $\operatorname{char} k \nmid |G|$. *Then the center of A is a symmetric algebra.*

5. N. Istrati, A. Otiman, *De Rham and twisted cohomology of Oeljeklaus-Toma manifolds* - preprint 2017.

The construction of Oeljeklaus-Toma (OT) manifolds arises from specific number fields. They are complex non-Kähler manifolds, which represent higher dimension analogues of Inoue surfaces $\mathcal{S}^0$. In this note, we compute their de Rham cohomology in terms of invariants associated to the background number field. This is done by two distinct approaches, one using invariant cohomology and the other one using the Leray-Serre spectral sequence. Moreover, we compute also their twisted cohomology, which has a particular interest for the OT manifolds admitting a locally conformally Kähler metric. We focus on this last class of OT manifolds, which has proved to be a good ground for testing existing cohomological conjectures in locally conformally Kähler geometry. In particular, we show that there is only one possible Lee class for LCK metrics.

6. C. Gherghe: *On a Yang-Mills type functional* - preprint 2017.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then the graph of u

$$G_u = \{(x, z) \in \mathbb{R}^{n+1} \ \mid \ z = u(x), x \in \Omega\},$$

is a minimal hypersurface if and only if satisfies the following differential equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{1}$$

In 1970 Calabi, in a paper in which he studied examples of Bernstein problems, noticed that if $n = 2$, u is a F-harmonic map, $F(t) = \sqrt{1 + 2t} - 1$, that is u is a critical point of the following functional:

$$E_F(u) = \int_{\mathbb{R}^2} F\left(\frac{\|du\|^2}{2}\right) \vartheta_g,$$

with respect to any compactly supported variation, $\|du\|^2$ being the Hilbert-Schmidt norm.

Following the ideas of Calabi, Yang and then Sibner showed that for $n = 3$, the equation (1) is equivalent, over a simply connected domain, to the vector equation

$$\nabla \times \left(\frac{\nabla \times A}{\sqrt{1 + |\nabla \times A|^2}}\right) = 0,$$

which arises in the nonlinear electromagnetic theory of Born and Infeld. Here A is a vector field in $\mathbb{R}^3$ and $\nabla \times (\,\cdot\,)$ is the curl of $(\,\cdot\,)$. Born-Infeld theory is of contemporary interest due to its relevance in string theory.

This observation leads Yang to give a generalized treatment of the equation (1), expressed in terms of differential forms, as follows

$$\delta\left(\frac{d\omega}{\sqrt{1+\|d\omega\|^2}}\right)=0, \tag{2}$$

for any $\omega \in A^p(\mathbb{R}^4)$. It is not very difficult to verify that the solution of equation (2) is a critical point of the following integral

$$\int_{\mathbb{R}^4}(\sqrt{1+\|d\omega\|^2}-1)\vartheta_g.$$

These facts gives as the motivation to study a similar functional, defined more general on Riemannian manifolds, functional which is on its definition, in some sense, similar to the well-known Yang-Mills functional. This new functional $YM_{BI}:\mathcal{C}(E)\to\mathbb{R}$ is defined by

$$YM_{BI}(D)=\int_M(\sqrt{1+\|R^D\|^2}-1)\vartheta_g.$$

In this paper we first derive the Euler-Lagrange equations and give an existence result. Section 3 is devoted to a conservation law of the functional. Finally in Section 4 we derive the second variation formula.

It is noteworthy that the dissemination of results was performed not only by publishing articles, but also by talks at international conferences and in departmental seminars:

1. F. Belgun: *Group actions on lcK manifolds*, Universität Hamburg, 13.11.2017.
2. A. Otiman: *Twisted cohomology of LCS solvmanifolds*, Cortona School "Kähler-Einstein metrics", Cortona, Italy, the 2nd of May, 2017.
3. A. Otiman: *Twisted cohomology of LCK manifolds*, Marburg School "Lie theory, PDE's and Geometry", Marburg, Germany, the 11th of October, 2017.
4. A. Otiman: *Twisted cohomology of LCK manifolds*, Global Analysis Seminar, University of Regensburg, Germany, the 25th of October, 2017.
5. G.E. Vîlcu: *Special classes of submanifolds in quaternionic-like geometries*, Annual conference of the Southern Africa Mathematical Sciences Association - SAMSA2017, November 20-24, 2017, Arusha, Tanzania.

Project manager,

Prof. dr. Liviu Ornea