HOPF ALGEBRA ACTIONS AND TRANSFER OF FROBENIUS AND SYMMETRIC PROPERTIES

S. DĂSCĂLESCU, C. NĂSTĂSESCU and L. NĂSTĂSESCU

Abstract

If H is a finite dimensional Hopf algebra acting on a finite dimensional algebra A, we investigate the transfer of the Frobenius and symmetric properties through the algebra extensions $A^H \subset A \subset A\#H$.
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1. Introduction and preliminaries

Let H be a finite dimensional Hopf algebra over a field k, and let A be an algebra on which H acts. In other words, A is a left H-module algebra, meaning that the algebra A also has a structure of a left H-module (with the action of $h \in H$ on $a \in A$ denoted by $h \cdot a$), such that $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1_A = \varepsilon(h) 1_A$ for any $h \in H$, $a, b \in A$; we use here the standard sigma notation for Hopf algebras, and ε is the counit of H. Two algebras are associated with such an action: the subalgebra A^H of invariants, consisting of all elements $a \in A$ such that $h \cdot a = \varepsilon(h)a$ for any $h \in H$, and the smash product $A\#H$, which is just $A \otimes H$ as a vector space, with $a \otimes h$ denoted by $a\#h$, endowed with an algebra structure with multiplication given by $(a\#h)(b\#g) = \sum a(h_1 \cdot b)\#h_2 g$.

A^H is a subalgebra of A, and A embeds in $A\#H$ by $a \mapsto a\#1$. Thus we have the algebra extensions $A^H \subset A \subset A\#H$. A general problem that can be posed is to study the transfer of a certain property through these extensions. There are several fundamental problems in Ring Theory and Representation Theory that can be formulated in this way. Indeed, it is enough to mention the following relevant examples of Hopf algebra actions; details on the first two can be found in [11], while for the third one we refer to [2].

(1) A finite group G acting as automorphisms on an algebra A defines an action of the group Hopf algebra kG on A. In this case A^{kG} is just the subalgebra A^G of fixed elements, and the smash product $A\#kG$ is the skew group ring $A * G$.

(2) If A is an algebra, and G is a finite group with neutral element e, then a

G-grading on A is just an action of $(kG)^*$, the dual Hopf algebra of kG, on A. In this case, the associated subalgebra of invariants is just the homogeneous component of degree e of A, and $A\#(kG)^*$ is the well-known graded smash product. (3) Let L be a finite dimensional restricted Lie algebra over a field of characteristic $p > 0$. Then the restricted universal enveloping algebra $u(L)$ has a Hopf algebra structure, and an action of $u(L)$ on an algebra A is just an action of L by derivations on A, which is compatible with the p-operation of L. The subalgebra $A^{u(L)}$ of invariants is just the subalgebra A^L of constants with respect to the action of L.

In this note we discuss the transfer of the Frobenius and symmetric properties through these extensions. We recall that a finite dimensional k-algebra is called Frobenius if A and its dual space A^* are isomorphic as left (or equivalently right) A-modules. If moreover, A and A^* are isomorphic as $A-A$-bimodules, then A is called a symmetric algebra. Frobenius algebras and symmetric algebras have a rich representation theory, and they occur in representation theory of groups, in quantum group theory, in the theory of compact oriented manifolds, in topological quantum field theory, etc., see [9]. It was proved in [3], and with a different approach in [7], that A is Frobenius if and only if so is $A\#H$. We give a new short proof of this result. We also give examples to show that except the Frobenius property in the extension $A \subset A\#H$, there is in general no other transfer of Frobenius or symmetric properties through any of the extensions $A^H \subset A$, $A \subset A\#H$ and $A^H \subset A\#H$. However, under certain conditions, the symmetric property transfers between A^H and $A\#H$.

We refer to [11] for notation and terminology about Hopf algebra actions. All algebras we work with are over a field k.

2. Some transfer results

We first prove the following general result.

Proposition 2.1. *Let B be a Frobenius algebra and A a subalgebra of B such that the left A-module B and the right A-module B are free. Then A is a Frobenius algebra.*

Proof. We first note that the left A-module B and the right A-module B have bases of the same finite cardinality. Indeed, $\dim_k B = n\dim_k A$, where n is the number of elements of a basis of the left A-module B. Similarly, $\dim_k B = m\dim_k A$, where m is the number of elements of a basis of the right A-module B. It follows that $m = n$, thus $B \simeq A^n$ as left A-modules, and also $B \simeq A^n$ as right A-modules. Since B is Frobenius, we have $B \simeq B^*$ as left B-modules, so then also as left A-modules. Since $B \simeq A^n$ as right A-modules, we get that

$B^* \simeq (A^*)^n$ as left A-modules. We obtain that $A^n \simeq (A^*)^n$ as left A-modules, and using the Krull-Schmidt Theorem we obtain that $A \simeq A^*$ as left A-modules. We conclude that A is a Frobenius algebra.

A first application is a very short proof of a result from [3].

COROLLARY 2.2. *Let H be a finite dimensional Hopf algebra acting on the finite dimensional algebra A. Then A is Frobenius if and only if so is $A\#H$.*

PROOF. It is known that $A\#H$ is a free left A-module and a free right A-module with bases of cardinality $dim(H)$. Then by Proposition 2.1, if $A\#H$ is Frobenius, then A is Frobenius.
On the other hand, if A is Frobenius, then so is $M_n(A) \simeq A \otimes M_n(k)$, since a tensor product of Frobenius algebras is Frobenius. The dual Hopf algebra H^* acts on $A\#H$ and the duality theorem says that $(A\#H)\#H^* \simeq M_n(A)$, see [11, Corollary 9.4.7]. Thus $(A\#H)\#H^*$ is Frobenius, and then so is $A\#H$.

We noticed that if A is Frobenius, then so is the matrix algebra $M_n(A)$. Since $M_n(A)$ is a free as a left A-module, and also as a right A-module, the following result, proved with a different method in [7], is an immediate consequence of Proposition 2.1.

COROLLARY 2.3. *Let A be a finite dimensional algebra and let n be a positive integer. If $M_n(A)$ is Frobenius, then so is A.*

We recall that a Morita context connecting two rings R and S is a sextuple (R, S, M, N, f, g) such that M is an $R-S$-bimodule, N is an $S-R$-bimodule, $f : M \otimes_S N \to R$ is an $R-R$-bimodule morphism, and $g : N \otimes_R M \to S$ is an $S-S$-bimodule morphism, such that $f(m \otimes n)m' = mg(n \otimes m')$ and $g(n \otimes m)n' = nf(m \otimes n')$ for any $m, m' \in M$ and $n, n' \in N$.

PROPOSITION 2.4. *Let (R, S, M, N, f, g) be a Morita context connecting the finite dimensional algebras R and S. If S is symmetric and f is surjective, then R is also symmetric.*

PROOF. Since f is surjective, then M is a finitely generated projective right S-module, and $R \simeq End(M_S)$, see [1, Exercise 5, page 266]. Then R is symmetric by [10, Exercise 25, page 456].

Let H be a finite dimensional Hopf algebra acting on the algebra A. Let $(e_i)_i$ be a basis of H, and let $(e_i^*)_i$ be the dual basis of H^*. The extension A/A^H is called H^*-Galois if the map

$$\rho : A \otimes_{A^H} A \to A \otimes_k H^*,\ \rho(a \otimes b) = \sum_i a(e_i \cdot b) \otimes e_i^*$$

is bijective. Note that the definition of ρ does not depend on the choice of the basis $(e_i)_i$. In the special case where H is the Hopf group algebra kG of a finite group G, H^*-Galois extensions include classical Galois field extensions, as well as Galois extensions in the case of finite groups acting on commutative rings. If $H = (kG)^*$, G a finite group, thus A is a G-graded algebra, then A/A_e is kG-Galois if and only if A is strongly graded.

We say that A has an element of trace 1 if there exists $a \in A$ such that $t \cdot a = 1$, where t is a non-zero left integral in H. Note that $(kG)^*$-module algebras, i.e. G-graded algebras, always have an element of trace 1.

If the finite dimensional Hopf algebra H acts on the algebra A, then by [11, Theorem 4.5.3] there is a Morita context connecting the rings A^H and $A\#H$. In the case where A/A^H is H^*-Galois, the Morita map to $A\#H$ is surjective. If A has an element of trace 1, then the other Morita map, to A^H, is surjective. Now we obtain directly from Proposition 2.4 the following two results.

Corollary 2.5. *Let H be a finite dimensional Hopf algebra acting on the finite dimensional algebra A such that A/A^H is left H^*-Galois. If A^H is symmetric, then so is $A\#H$.*

Corollary 2.6. *Let H be a finite dimensional Hopf algebra acting on the finite dimensional algebra A such that A has an element of trace 1. If $A\#H$ is symmetric, then so is A^H.*

Examples of symmetric algebras with non-symmetric centers were given in [8]. A positive result was proved in the same paper, by showing that a finite dimensional G-graded division algebra is always symmetric, and if $\operatorname{char} k$ does not divide $|G|$, then its center is a symmetric algebra. As another consequence of Proposition 2.1, we have the following.

Corollary 2.7. *Let A be a symmetric algebra which is a free module over its center $Cen(R)$. Then $Cen(R)$ is a symmetric algebra.*

3. Examples

We provide examples to show that for a general action, no transfer of Frobenius or symmetric properties takes place between any pair of algebras among A^H, A and $A\#H$, except the one indicated in Corollary 2.2. We will use an equivalent characterization of Frobenius algebras: A is Frobenius if it has a hyperplane which does not contain any non-zero left ideal, see [10, Theorem 3.15]. Also, A is symmetric if and only if it has a hyperplane $\mathcal{H}$ which does not contain any non-zero left ideal, and such that $[A, A] \subset \mathcal{H}$ see [10, Theorem 16.54]. Here $[A, A]$ is the subspace of A spanned by all commutators $[a, b] = ab - ba$, with

$a, b \in A$. Obviously, for commutative algebras, the symmetric and the Frobenius properties coincide.

EXAMPLE 3.1. A Frobenius (symmetric) $\not\Rightarrow A^H$ Frobenius (symmetric)
We provide two such examples.
(1) Let k be a field of characteristic 2. Let A be the k-algebra generated by x, y with relations $x^2 = y^2 = 0, xy = yx$. A has a basis $\{1, x, y, xy\}$. Then A is a Frobenius algebra, thus also symmetric, since A is commutative. Indeed, the hyperplane $< 1, x, y >$ of A does not contain non-zero ideals of A, see [10, Example 3.15B].

Let σ be the algebra automorphism of A such that $\sigma(x) = y$ and $\sigma(y) = x$. Then σ has order 2, so it induces an action of the cyclic group C_2 as automorphisms on A. The associated subalgebra of invariants is

$$A^{kC_2} = A^\sigma = \{\alpha 1 + \beta x + \beta y + \gamma xy \mid \alpha, \beta, \gamma \in k\} =< 1, x + y, xy >$$

Denote $p = x + y$ and $q = xy$. Then A^σ is the algebra generated by p, q, subject to relations $p^2 = q^2 = pq = qp = 0$ (here is where we need characteristic 2).

We claim that A^σ is not symmetric. Indeed, we first note that the one dimensional subspace spanned by an element of the form $\beta p + \gamma q$ is an ideal of A^σ. If $\mathcal{G}$ is a hyperplane of A^σ, let $\{\alpha_1 1 + \beta_1 p + \gamma_1 q, \alpha_2 1 + \beta_2 p + \gamma_2 q\}$ be a basis of $\mathcal{G}$. If $\alpha_1 = 0$ or $\alpha_2 = 0$, then $\mathcal{G}$ clearly contains a non-zero ideal. If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, take a non-zero linear combination of the two basis elements, with a zero coefficient for 1, and again obtain a non-zero ideal inside $\mathcal{G}$. We conclude that any hyperplane of A^σ contains a non-zero ideal, so A^σ is not symmetric.

The referee pointed out the following shorter and more conceptual proof of the fact that A^σ is not symmetric. We see that A^σ is a local algebra with Loewy length 2, and if it were Frobenius it would have simple socle. But A^σ has length 3 (= the dimension), so the socle has length (and dimension) 2.
(2) Let B be a finite dimensional algebra which is not Frobenius, and let A be the trivial extension of B. More precisely, $A = B \oplus B^*$, with multiplication defined by $(b, f)(b', f') = (bb', bf' + fb')$. Then A is a symmetric algebra, see [10, Example 16.60]. On the other hand, if $C_2 =< c >= \{e, c\}$ is the cyclic group of order 2, then A is a C_2-graded algebra with the homogeneous components $A_e = B \oplus 0 \simeq B$, and $A_c = 0 \oplus B^*$. Then A is a symmetric algebra on which the Hopf algebra $(kC_2)^*$ acts, while $A^{(kC_2)^*} = A_e$ is not even Frobenius.

EXAMPLE 3.2. A^H Frobenius (symmetric) $\not\Rightarrow A$ Frobenius (symmetric)
We first note that this is expected, since A^H may be really small, even the field itself (so almost trivially Frobenius), while A is not. We present two examples.
(1) Let A be the algebra of upper triangular $n \times n$-matrices over k, where $n \geq 2$. Let G be a finite group with neutral element e, such that there exist

$g_1, \ldots, g_{n-1} \in G$ with the property that $\prod_{i \leq p \leq j} g_p \neq e$ for any $1 \leq i \leq j \leq n-1$. One can find such elements recurrently, by starting with a non-trivial g_1, then by choosing at step j some $g_j \in G \setminus \{e\}$ different from $(g_i \ldots g_{j-1})^{-1}$ for any $1 \leq i \leq j-1$. It is clear that such a choice is possible if G has enough elements (for instance at least n elements). Define a G-grading on the algebra A, by setting the matrix units $e_{11}, \ldots, e_{nn}$ to be homogeneous of degree e, and e_{ij} to be homogeneous of degree $g_i \ldots g_{j-1}$ for any $1 \leq i < j \leq n$. Then the homogeneous component of degree e of A is isomorphic to k^n, which is semisimple, thus symmetric. However, A is not Frobenius; see for example [6], where it is proved that a structural matrix algebra over k is Frobenius only when it is a product of diagonal blocks which are full matrix algebras. We conclude that A is a left $(kG)^*$-module algebra, and $A^{(kG)^*} = A_e$ is symmetric, while A is not even Frobenius. Other examples can be obtained by taking A to be a more complicated (non-semisimple) structural matrix algebra.

The simplest such example is $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ with the $C_2 =< c >$-graded algebra structure

$$A_e = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad A_c = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}.$$

(2) This example was given by the referee. Let k be a field of positive characteristic p, and let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, which is not Frobenius. The Jordan block $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ determines an inner automorphism $\sigma : A \to A, \sigma(X) = J^{-1}XJ$ for any $X \in A$. We have $\sigma(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} a & b+a-c \\ 0 & c \end{pmatrix}$. Then σ has order p, and thus it induces an action of the cyclic group C_p on A. The subalgebra of invariants is $A^{C_p} = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in k\}$, which is isomorphic to $k[X]/(X^2)$, a symmetric algebra.

Example 3.3. A symmetric $\not\Rightarrow$ $A\#H$ symmetric
Let H be a finite dimensional Hopf algebra acting on the finite dimensional algebra A, such that A is symmetric, but A^H is not symmetric; examples of such actions are given in Example 3.1. Also assume that A has an element of trace 1. This condition is always satisfied in the case where H is semisimple. Then [11, Lemma 4.3.4] shows that there exists an idempotent $e \in A\#H$ such that $e(A\#H)e \simeq A^H$ as algebras. Then $A\#H$ is not symmetric, otherwise $e(A\#H)e$ is symmetric by [10, Exercise 25, page 456], and then so is A^H, a contradiction.

Example 3.4. $A\#H$ symmetric $\not\Rightarrow$ A symmetric
Let H be a finite dimensional Hopf algebra acting on the finite dimensional

algebra B, such that B is symmetric, but $B\#H$ is not symmetric; such examples are given in Example 3.3. Then $A = B\#H$ is not symmetric, A is a left H^*-module algebra, and $A\#H^* = (B\#H)\#H^* \simeq M_n(B)$ by the duality theorem. Since B is symmetric, then so is $M_n(B)$. Thus $A\#H^*$ is symmetric.

Example 3.5. $A\#H$ Frobenius (symmetric) $\not\Rightarrow$ A^H Frobenius (symmetric) Let A and $H = kC_2$ be as in Example 3.1 (1). We have seen that A^H is not Frobenius. We show that $A\#H = A * C_2$ is symmetric.

We first compute $[A * C_2, A * C_2]$. If $a, b \in A$, then $[ae, b\sigma] = (ab + \sigma(a)b)\sigma$, and a simple computation shows that $[Ae, A\sigma] =< (x + y)\sigma, xy\sigma >$. On the other hand, $[a\sigma, b\sigma] = (a\sigma(b) + \sigma(a)b)e$ for any $a, b \in A$, so again we easily get $[A\sigma, A\sigma] =< (x + y)e >$. Clearly $[Ae, Ae] = 0$. We obtain that

$$[A * C_2, A * C_2] =< (x + y)\sigma, xy\sigma, (x + y)e >$$

Let $\mathcal{H} =< e, xe, ye > +A\sigma$, which is a hyperplane in $A * C_2$. Clearly $[A * C_2, A * C_2] \subset \mathcal{H}$.

We show that $\mathcal{H}$ does not contain any non-zero left ideal of $A * C_2$. Indeed, assume that $(A*C_2)u \subset \mathcal{H}$ for some $u \in A*C_2$. Write u as a linear combination of the basis $\{e, xe, ye, xye, \sigma, x\sigma, y\sigma, xy\sigma\}$. Since $u \in \mathcal{H}$, the coefficient of xye must be zero. If the coefficient of e is non-zero, then since $(xye)e = xye$, and in $(xye)u$ the element xye can appear only from the multiplication $(xye)e$, we have that the coefficient of xye in $(xye)u$ is non-zero, thus $(xye)u \notin \mathcal{H}$, a contradiction. Similarly, the relations

$$(ye)(xe) = xye,\ (xe)(ye) = xye,\ (xy\sigma)\sigma = xye,$$

$$(x\sigma)(x\sigma) = xye,\ (y\sigma)(y\sigma) = xye,\ \sigma(xy\sigma) = xye$$

show that the coefficients of $xe, ye, \sigma, x\sigma, y\sigma, xy\sigma$ in u must be zero. These show that $u = 0$.

We conclude that $A * C_2$ is symmetric.

Example 3.6. A^H Frobenius (symmetric) $\not\Rightarrow$ $A\#H$ Frobenius (symmetric) We take A and H as in Example 3.2. Then A^H is symmetric, while A is not Frobenius. Then $A\#H$ is not Frobenius either, by Corollary 2.2.

Regarding the Frobenius notion, another interesting question is whether the extensions $A^H \subset A$, $A \subset A\#H$ and $A^H \subset A\#H$ are Frobenius extensions of rings. We recall that a ring extension $R \subset S$ is Frobenius if S is a finitely generated projective left R-module and $S \simeq Hom_R(S, R)$ as $S - R$-bimodules, see [4] or [5] for details.

It is known that $A \subset A\#H$ is a Frobenius extension, see for instance [5, Corollary 27].

The extension $A^H \subset A$ is not Frobenius in general. Indeed, if A is a G-graded algebra, where G is a finite group, such that A is not projective as a left A_e-module (where e is the neutral element of G), then A is a left H-module algebra, where $H = (kG)^*$, the dual of the group Hopf algebra, $A^H = A_e$, and obviously $A^H \subset A$ is not a Frobenius extension. There are many examples of graded rings A which are not projective over A_e. A simple such example is obtained by taking a commutative ring R and an R-module M which is not projective, and considering the trivial extension $A = R \oplus M$, which is a ring with the multiplication $(r, m)(r', m') = (rr', rm' + r'm)$ for any $r, r' \in R, m, m' \in M$. Moreover, A is graded by the cyclic group $C_2 =< \sigma >$ of order 2 with $A_e = R \oplus 0, A_\sigma = 0 \oplus M$, and clearly A is not A_e-projective.

Finally, $A^H \subset A\#H$ is also not Frobenius in general. Indeed $A\#H \simeq A^n$ as left A-modules, where n is the dimension of H. This is also an isomorphism of left A^H-modules, so in the case where A is not projective as a left A^H-module (as in the previous example), we see that $A\#H$ is also not A^H-projective.

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Faculty of Mathematics and Computer Science
University of Bucharest
Str. Academiei 14
Bucharest, RO-010014
Romania
E-mail: sdascal@fmi.unibuc.ro

Institute of Mathematics of the Romanian Academy
PO-Box 1-764
Bucharest, RO-014700
Romania
E-mail: lauranastasescu@gmail.com

Institute of Mathematics of the Romanian Academy
PO-Box 1-764
Bucharest, RO-014700
Romania
E-mail: Constantin_nastasescu@yahoo.com